Chapter 12. Basic Equations of Stellar Structure

To understand the H-R diagram we must first understand the basic structure of a star and then how they evolve. Prior to 1905, it was not understood how a star could power itself. That is, the luminosity of the Sun is so large that it could not last for more than about 20 million years on the basis of the gravitational potential energy that it gained by contracting to its present size. Prior to 1905, therefore, astronomers and physicists believed that the Earth could not have been around form more than 20 million years or so and there certainly could not have been life on this planet for longer than that, since life requires sunlight to exist. The geologists and biologists of that era were sure, however, that the slow processes of geologic and biologic evolution required billions of years to occur, not millions, so there was a problem!

A solution became possible in 1905 when Einstein wrote down his famous equation, $E=mc^2$. It became at least theoretically possible that stars could convert some of their own mass into energy and gain the power needed to keep themselves shining for billions of years. It took another 35 years before physicists worked out the details of this process, namely the p-p reactions that turn Hydrogen into Helium, releasing energy in the process. Now we understand a star as basically a nuclear fusion machine that produces energy by fusion in its core to replace the energy it radiates at its surface. The Sun has so much Hydrogen fuel that it can go on doing this for about 10 billion years before significantly depleting the H in its core.

The basic theory of stellar structure assumes spherical symmetry, so that all variables depend on only one thing, the distance (r) from the center of the star. On spherical shells of radius r, all physical variables (e.g. temperature, density, pressure, chemical composition, etc.) are assumed to be uniform. Note that this assumption should be very good for the Sun (and most stars) since it spins very slowly and is spherically symmetric at its surface to such a high degree of accuracy that we cannot measure any departure from a sphere. Some rapidly rotating stars (generally young ones) may not quite fit this assumption and more sophisticated models may be required to explain some of their properties. That is beyond the scope of this course!

We have learned over the years that a star can be well described by only 4 differential equations that are based on certain physical principles, plus 3 “auxiliary” equations that relate some of the variables. The principle variables of stellar structure are pressure (P), temperature (T), density ($\rho$), luminosity through a shell at r ($L_r$) and mass interior to r ($M_r$). Auxiliary parameters include the rate of nuclear energy generation and the opacity of the gas (i.e. how transparent it is to radiation). Free parameters for the models include the chemical composition of the initial gas and the initial mass that came together to form
the star. Complicating factors such as the magnetic field are generally ignored in making the models, although it can play a role. Another complicating factor is that stars are often highly convective (i.e. they have streams of hotter gas that moves like a warm ocean current carrying energy from one place to another). Convection and magnetic effects are very hard to model and are generally treated in highly simplified ways, if at all.

Despite the simplifications, the theory of stellar structure has had good success at providing us with a basic understanding of how stars evolve and why the H-R diagram looks the way it does (e.g. what the main sequence is, what red giants are, etc.). In the basic theory there are only four equations and we can derive three of them here. The fourth is a bit more difficult to deal with and is left for an upper level course in stellar structure, although its nature will be described here.

1. Mass interior to \( r \) (\( M_r \))

The first equation of stellar structure comes from the principle of conservation of mass. Of course, stars do not precisely conserve their mass – they turn some of it into energy by nuclear fusion. But that amount is so small that we can ignore it and use the principle of mass conservation as if no nuclear processes occurred. Consider a shell inside the star that has radius \( r \) and a thickness \( dr \) (indicating that it is a very thin shell – of infinitesimally small thickness). Suppose that the density in that shell is \( \rho(r) \) and recall that it is uniform everywhere in the shell by the spherical symmetry assumption. The volume \( (dV) \) of such a shell is \( dV = 4\pi r^2 dr \) (i.e. its surface area times its thickness). The mass of such a shell is \( \rho(r)dV = 4\pi r^2 \rho dr \). Now, by the conservation of mass we can write that the mass interior to \( r \) must increase by an amount \( (dM_r) \) equal to the mass of the shell at \( r \), i.e.

\[
dM_r = 4\pi r^2 \rho dr
\]

or, as it is usually written,

\[
\frac{dM_r}{dr} = 4\pi r^2 \rho.
\]

This is the first equation of stellar structure.

2. Conservation of Energy

A very similar equation comes from the principle of conservation of energy. While we can ignore the mass loss that comes from nuclear fusion, we cannot ignore the energy produced, which is enormous. Here we need a parameter that describes how much energy
is created in each shell and that parameter is usually called $\epsilon$. Its value depends on the chemical composition in the shell (assumed uniform), as well as the density and temperature. Calculating $\epsilon$ is very difficult and involves calculating the rates of all the nuclear reactions that occur within the shell to contribute to the energy. We will discuss some important nuclear processes later. Here we simply wish to write the basic equation. If $\epsilon$ is defined as the amount of power produced per gram within the shell then multiplying the mass of the shell by $\epsilon$ gives the total addition to the power of the star, which we call $dL_r$. Hence, conservation of energy requires that

$$dL_r = 4\pi r^2 \rho \epsilon dr$$

or, as it is usually written

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon.$$

This is the second equation of stellar structure.

3. Equation of Hydrostatic Equilibrium

The physical principle of the next equation is the same principle that is used to calculate how the pressure in a lake or ocean increases as one goes deeper into it. Therefore, it is referred to as hydrostatic equilibrium. Physically it states that the pressure in a fluid (including a gas) must increase in order to support the weight of the column of fluid above it. (The same principle is used to calculate air pressure as a function of height above sea level.) The pressure gradient (acting upwards) must be just right to balance the force of gravity acting downwards to keep each layer in a fluid at its static level.

Considering, again, a shell of radius $r$ inside the star, we know that the gravitational force acting on that shell is

$$F_{grav} = \frac{GM_r m_{shell}}{r^2}$$

but plugging in for the mass of the shell, $4\pi r^2 \rho dr$, we have

$$F_{grav} = \frac{GM_r 4\pi r^2 \rho dr}{r^2}$$

Now, the counter-balancing force is the fact that the pressure at $r$, $P(r)$, must be greater than the pressure at little further out in the star $P(r+dr)$. Since pressure is force per unit area to get the full pressure difference on the shell we must multiply by the surface area of
the shell at radius r, which is \(4\pi r^2\). So the outward directed force coming from the pressure gradient (i.e. difference in pressure at r and at r+dr) is

\[ F_{\text{pressure gradient}} = (P(r) - P(r + dr))4\pi r^2 \]

Now, setting the pressure gradient force equal to the gravitational force (and recalling that they act in opposite directions!) we have:

\[ \frac{P(r) - P(r + dr)}{dr} = \frac{GM_r \rho}{r^2} \]

Taking the limit as dr goes to zero means that the left hand side becomes the negative of the derivative of P, so the equation of hydrostatic equilibrium is:

\[ \frac{dP}{dr} = -\frac{GM_r \rho}{r^2} \]

This is the third equation of stellar structure

4. Energy Transport Equation

The fourth and final equation of stellar structure is set by how a star transports its energy from the inside, where it created by nuclear fusion, to the outside where it is radiated into space as photons. There are two basic methods for transporting energy in most stars: radiation and convection. Which occurs depends largely on the opacity of the matter. In regions of the star where it is fairly transparent the energy can travel by photons. This proceeds like a random walk. A photon only goes, on average, about 1 cm before it is absorbed and re-emitted by matter. It can take millions of years for a photon to diffuse out of the star by this random walk process. If we were to turn off the nuclear power of the Sun today, it would continue to shine at the current rate for millions of years into the future just on the basis of the photons that are already working their way out!

A star that is getting its energy out by radiation is said to be using radiative transport. Sometimes, however, the energy input by nuclear reactions is so rapid or the material of the star is so opaque that the energy cannot get out by radiation fast enough. In that case the temperature and pressure in the interior of the star tends to build up rapidly and causes the matter in some zone to be over-pressured. That element of a star then becomes buoyant and tends to easily float up towards the surface. Soon, a current can develop with warm material flowing upwards and being replaced by colder material (it’s still very hot of course!) flowing downwards. One gets a “convection current” developing, much like boiling water.
This is a very effective heat transport mechanism. The star basically boils. The Sun does this in its outer one-third or so of its mass. Convection is important in mixing a star and also helps generate strong magnetic fields, which erupt at the surface of the sun as sunspots. This drives solar flares and the sunspot cycle and the solar wind.

The equations governing radiative or convective transport are not so simple as the first three equations of stellar structure and we will save them for an upper level course in stellar astrophysics. In some cases, there is a third mechanism of energy transport, namely conduction. This occurs when atoms are locked in place within a solid ...not something that usually occurs in a star. However, as we will see later, very dense cores within stars can become essentially solid, (actually electron-degenerate is the term for this condition) and while the atoms have the freedom of a gas, the electrons are locked in position like a solid and can easily transmit energy via conduction. This is the preferred mechanism of energy transport for very dense, degenerate cores of stars and for white dwarf stars (and neutron stars).

5. The Kelvin Time

Before turning to the important question of nuclear fusion, let us see exactly why it is required for a star to have a long life time (long by astronomical standards – most people would consider 20 million years to be long, but not astronomers!) We can estimate the total gravitational potential energy of the Sun in the following way. It is a sphere and we know that its average density is $\langle \rho \rangle = \frac{M}{V} = 1.4 \text{ gm cm}^{-3}$. We can calculate the contribution of each shell within it to its gravitational potential energy if we recall that for two masses, $M$ and $m$ separated by a distance $r$, the gravitational potential energy is given by $\frac{GMm}{r}$. Replacing $M$ with $Mr$ (because it is only the mass interior to $r$ that pulls on the shell at $r$, and replacing $m$ with the mass of the shell (its density times its volume, as before) and integrating from the center of the star ($r=0$) to its surface ($r=R$; we use $R$ as the radius of the star), we have for the total gravitational potential energy of the star ($E_G$):

$$E_G = \int_0^R \frac{GM_r \pi r^2 \rho}{r} dr$$

In the case of uniform density, we can take $\rho = \langle \rho \rangle = \text{constant}$ and do the integration explicitly. We find that:

$$E_G = \frac{3GM^2}{5R}$$

According to the Virial Theorem, as a star forms, one-half of the gravitational potential
energy released goes into heating up the star and one-half is available to be radiated, producing its luminosity (L). Putting in solar values of the mass (M), Radius (R) and current luminosity (L) of the Sun, we can calculate the length of time ($t_{\text{Kelvin}}$), known as the Kelvin time, that the Sun could have radiated its current luminosity if it had no energy source other than gravitational contraction. For the Sun, we find:

$$t_{\text{Kelvin}} = \frac{3}{10} \frac{GM^2}{RL} \approx 10 \text{ million years.}$$

Today we recognize the Kelvin time as the approximate time that it takes the star to reach the main sequence stage, where it turns on its nuclear fusion in its core and stops the gravitational contraction, settling into a stable star (on the main sequence) that can last there for billions of years. The time in a star’s life before it has turned on its nuclear fusion, is known as its pre-main sequence lifetime, and the Kelvin time is a good estimate of the pre-main sequence lifetime of stars. Of course, our estimate above is a little bit off because the Sun is not really of uniform density...its interior is more highly compressed than its exterior, so it does have a little more gravitational potential energy than the simple calculation above would suggest. Its pre-main sequence lifetime is more like 30 million years than 10 million years, but this is a relatively small correction to astronomers (a factor of 3).