18 - POLYTROPIC AND ISOTHERMAL SPHERES

Polytropes

If the heat content of a gas is \( Q \), its internal energy is \( U \), its pressure is \( P \), and its volume is \( V \) (all per unit mass), then the change in its internal energy will be equal to the heat put into the system minus the work the system does:

\[
dU = dQ - PdV, \\
or \quad dQ = dU + PdV
\]

The ideal gas law states that \( PV = RT \) (remember \( V = \frac{1}{\rho} \), the "specific volume") where \( R \) is the gas constant (in appropriate units). Hence, \( PdV + VdP = RdT \).

Define the specific heat when the parameter \( \alpha \) is constant as

\[
C_\alpha = \left[ \frac{dQ}{dT} \right]_{\alpha=\text{constant}}
\]

There are 4 special cases of \( C \):

A. Adiabatic \( \Rightarrow C = 0, dQ = 0 \)
B. Isothermal \( \Rightarrow C = \infty, dT = 0 \)
C. Constant Volume \( C = C_V \)

\[
dQ = dU + PdV = \frac{dU}{dT}dT + PdV \\
C_V = \left( \frac{dQ}{dT} \right)_V = \frac{dU}{dT}
\]

D. Constant Pressure \( C = C_P \)

\[
dQ = dU + PdV = dU + RdT - VdP \\
C_P = \left( \frac{dQ}{dT} \right)_P = \frac{dU}{dT} + R \\
C_P = C_V + R
\]

For a monatomic ideal gas,

\[
C_V = \frac{3}{2} R, \quad C_P = \frac{5}{2} R \quad \text{and} \quad \frac{C_P}{C_V} \equiv \gamma = \frac{5}{3}
\]
Generally,

\[ dQ = dU + PdV \]

\[ CdT = C_v dT + PdV \]

but

\[ P = \frac{RT}{V} = \left(\frac{C_p - C_v}{C_p - C_v}\right)T \]

\[ CdT = C_v dT + \left(\frac{C_p - C_v}{C_p - C_v}\right) dV \]

\[ \left(\frac{C - C_v}{C_p - C_v}\right) \frac{dT}{T} = \frac{dV}{V} \quad \text{or} \quad \frac{dT}{T} = \left(\frac{C_p - C_v}{C - C_v}\right) \frac{dV}{V} \]

\[ \frac{dT}{T} + \left(\frac{C_p - C_v}{C_v - C}\right) \frac{dV}{V} = 0 \]

Define the effective \( \gamma' = \left(\frac{C_p - C}{C_v - C}\right) \) and a polytropic process as one where \( \gamma' = \text{const.} \)

\[ \frac{dT}{T} + \frac{C_p - C_v}{C_v - C} \frac{dV}{V} = 0 \]

\[ \frac{dT}{T} + \left[\frac{C_p - C}{C_v - C} - \frac{C_v - C}{C_v - C}\right] \frac{dV}{V} = 0 \]

\[ \frac{dT}{T} + (\gamma' - 1) \frac{dV}{V} = 0 \]

\[ d \ln T + (\gamma' - 1) d \ln V = 0 \]

\[ TV^{\gamma' - 1} = \text{const.} \]

But \( P = \frac{RT}{V} \quad \text{or} \quad T = \frac{PV}{R} \quad \text{or} \quad V = RTP^{-1} \) so,

\[ PV^{\gamma'} = \text{const.} \]

\[ PT^{\gamma'} P^{-\gamma'} = \text{const.} \]

\[ P^{1-\gamma'} T^{\gamma'} = \text{const.} \]
Note that if:

\[ C = 0 \Rightarrow \gamma' = \gamma = \frac{5}{3} \Rightarrow PV^{\gamma'} = \text{const. Adiabatic Ideal Gas Law} \]

\[ C = \infty \Rightarrow \gamma' = 1 \quad \Rightarrow TV^{\gamma'-1} = T = \text{const. Isothermal} \]

\[ C = C_p \Rightarrow \gamma' = 0 \quad \Rightarrow PV^{\gamma'} = P = \text{const. Constant Pressure} \]

\[ C = C_v \Rightarrow \frac{dV}{V} = \frac{C - C_v}{C_p - C_v} \frac{dT}{T} = 0 \quad \text{so } dV = 0 \quad \text{Constant Volume} \]

For arbitrary (non-ideal) gases, we can define

\[ PV^{\Gamma_1} = \text{const.} \]

\[ P^{1-\Gamma_2}T^{\Gamma_2} = \text{const.} \]

\[ TV^{\Gamma_3-1} = \text{const.} \]

For a mixture of gas and radiation, it can be shown (see Chandrasekhar’s Stellar Structure, pp. 55-59) that for adiabatic changes,

\[ \Gamma_1 = \beta + \frac{(4 - 3\beta)^2(\gamma - 1)}{\beta + 12(1 - \beta)(\gamma - 1)} \]

\[ \Gamma_2 = 1 + \frac{(4 - 3\beta)(\gamma - 1)}{\beta^2 + 3(\gamma - 1)(1 - \beta)(4 + \beta)} \]

\[ \Gamma_3 = 1 + \frac{(4 - 3\beta)(\gamma - 1)}{\beta + 12(\gamma - 1)(1 - \beta)} \]

For pure radiation, \( \beta = \frac{P_g}{P} = 0 \) so

\[ \Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{4}{3} \quad \Rightarrow \]

\[ \Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{4}{3} \]

For a monatomic gas, \( \left( \gamma = \frac{5}{3} \right) \) and no radiation \( \left( \beta = 1 \right) \), \( \Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{5}{3} \).

For \( 0 \leq \beta \leq 1 \), \( \Gamma_1 \neq \Gamma_2 \neq \Gamma_3 \).
A polytropic system is one where changes in $P$ and $V$ take place such that $P \propto V^{-\Gamma} \propto \rho^\Gamma$.

Let $P = K \rho^{\frac{n+1}{n}}$ where $n$=“polytropic index” = $\frac{1}{\Gamma-1}$.

From Eqs of HE and MC, $M(r) = -\frac{r^2}{\rho G} \frac{dP}{dr}$, $\frac{dM(r)}{dr} = 4\pi r^2 \rho$. We could integrate these if we had one more equation involving these variables ($P(r), M(r), \rho(r)$). We will use the polytropic relation to accomplish this task.

$$\frac{d}{dr} \left( r^2 \frac{dP}{\rho \, dr} \right) = -4\pi r^2 G \rho$$

**But for polytropes**, $P = K \rho^{\frac{n+1}{n}} = K \left( \rho^n \right)^{n+1}$

$$\frac{d}{dr} \left( r^2 \left( n+1 \right) \frac{d\left( \rho^n \right)}{dr} \right) = -4\pi r^2 G \rho$$

At this point, let us make a change of variables

$$\rho = \lambda \cdot \theta^n \quad (\theta = \theta(\xi))$$

$$r = \alpha \cdot \frac{\xi}{\pi}$$

This gives,

$$K \left( n+1 \right) \frac{1}{\alpha} \frac{d\xi}{d\theta} \left[ \alpha^2 \frac{d^2 \xi}{d\theta^2} \left( \lambda^2 \theta \right) \right] = -4\pi \alpha^2 \xi^2 G \lambda \theta^n$$

Or, by setting $\alpha = \left[ \frac{K \left( n+1 \right) \lambda^{\frac{1}{n}-1}}{4\pi G} \right]^{\frac{1}{2}}$, a dimensionless equation results:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left[ \xi^2 \frac{d\theta}{d\xi} \right] = -\theta^n$$

**Lane-Emden Equation**
Picking \( n \) specifies the type of physical system, while \( K \) gives the total mass (see later).

Boundary Conditions:

At the core:

\[
r \to 0 \quad \text{so} \quad \xi \to 0
\]

\[
\rho \to \rho_c \quad \theta \to 1 \text{ and } \lambda = \rho_c
\]

Also, from HE + MC,

\[
\frac{d}{dr} \left[ r^2 \frac{dP}{\rho \, dr} \right] = -4\pi r^2 G \rho \quad \leftarrow \text{previous page}
\]

\[
or \quad \frac{r \, d^2P}{\rho \, dr^2} + 2 \frac{dP}{\rho \, dr} - \frac{r \, dP}{\rho^2 \, dr \, dr} = -4\pi G \rho r
\]

as \( r \to 0 \), \( \frac{dP}{dr} \to 0 \) but \( P \propto \theta^{1+n} \), \( r \propto \xi \)

so at \( \xi = 0 \), \( \frac{d\theta}{d\xi} \to 0 \) \( \left( \text{i.e.,} \quad \frac{dP}{dr} \propto \frac{d(\theta^{1+n})}{d\xi} \to 0 \right) \)

Only 3 analytic solutions exist:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \theta(\xi) )</th>
<th>( \xi ) (where ( \theta ) first ( \to 0 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sqrt{6} )</td>
<td>( \frac{1-\xi^2}{6} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{\sin \xi}{\xi} )</td>
<td>( \pi )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{\sqrt{1+\xi^2/3}} )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

(Note: to solve for arbitrary \( n \), one can get a general series solution

\[
\theta_n = 1 - \frac{\xi^2}{6} + \frac{n}{120} \xi^4 - \frac{8n^2 - 5n}{15120} \xi^6 + \cdots
\]

but this so-called E(for Emden)-solution is true only for \( \xi \to 0 \).
Physical Variables

\[ R = \left( \frac{(n+1)K \rho_c^{1-n}}{4\pi G} \right)^{\frac{1}{n}} \]

\[ r = \left( \frac{(n+1)K \rho_c^{1-n}}{4\pi G} \right)^{\frac{1}{n}} \]

\[ M = -4\pi \left( \frac{(n+1)K}{4\pi G} \right)^{\frac{3}{2-n}} \left[ \frac{d\theta}{d\xi} \right]^{\frac{1}{n}} \]

\[ P_c = \frac{GM^2}{R^3} \left( 4\pi \left( n + 1 \right) \left[ \frac{d\theta}{d\xi} \right]^{\frac{2}{n}} \right)^{-1} \]

\[ \text{Binding Energy} \quad \Omega = -\frac{3}{5-n} \frac{GM^2}{R} \]

\[ \rho_c = \frac{\frac{-\xi}{3} \left[ \frac{d\theta}{d\xi} \right]^{\frac{1}{n}}}{\left( \frac{4\pi R^3}{\xi} \right)^{\frac{1}{n}}} \quad \text{(and } \rho = \rho_c \theta^n) \]

Note: \( K = \frac{k}{\mu m_H} \) "\( \theta \)" where "\( \theta \)" = const. = \( \frac{T}{\rho^{\gamma-1}} \) the "polytropic temperature"

Also: \( T_c = \frac{P_c \mu m_H}{\rho_c k} \) and \( T = T_c \theta \)

There is also a Mass-Radius Relation:

\[ K = \frac{1}{n+1} \left[ \frac{4\pi}{\xi^{n+1} \left[ \frac{d\theta}{d\xi} \right]^{n-1}} \right] \left[ \frac{G M^{n-1} R^{3-n}}{\pi} \right]^{\frac{1}{n}} \]

or, \( R \propto M^{\frac{1-n}{3-n}} \) for a certain family (n) of solutions.
Summary

Given $M$, $R$, $n$, calculate the internal parameters

$$\rho_c = \frac{\xi_1}{3} \left[ \frac{M}{\frac{4}{3} \pi R^3} \right] \quad \text{and} \quad \rho = \rho_c \theta^n$$

$$P_c = \frac{GM^2}{R^4} \left[ 4\pi (n+1) \left( \frac{d\theta}{d\xi} \right) \right]^{2n-1} \quad \text{and} \quad P = P_c \theta^{n+1}$$

$$T_c = \frac{P_c}{\rho_c} \frac{\mu m_H}{k} \quad \text{and} \quad T = T_c \theta$$

$$K = \left[ \frac{4\pi}{n+1} \left( \frac{d\theta}{d\xi} \right) \right]^{n-1} \quad \text{and} \quad G M^{\frac{n-1}{n}} R^\frac{3-n}{n}$$

$$r = \left[ \frac{(n+1)K}{4\pi G} \rho_c^{\frac{1-n}{n}} \right]^{\frac{1}{n}}$$

$$M(\xi) = -4\pi \left[ \frac{(n+1)K}{4\pi G} \rho_c^{\frac{3-n}{n}} \xi^2 \frac{d\theta}{d\xi} \right]^{\frac{1}{n}}$$

For a star with radiation pressure and $\beta = \text{constant}$,

$$P_{g+r} = \left[ \frac{1-\beta}{\beta^4} \left( \frac{k}{\mu m_H} \right)^{\frac{4}{3}} \frac{3}{a} \right]^{\frac{1}{n}} \rho^{\frac{4}{3}} \propto \rho^{\frac{4}{3}}$$

Such a star therefore has $\gamma = \frac{4}{3}$ and is a polytrope of index $n=3$.

We will also see later on that a fully convective star behaves like a polytrope of index $n=1.5$. Also, many radiative envelopes of stars are nearly polytropic with $n=3.20-3.25$. 
While none of these solutions can be obtained analytically, models of $n=1.5$ and $n=3$ can be obtained by numerical techniques, and the results are of great use in getting approximate models of stars.

And while it is true that, as the speed of computers continues to increase rapidly, making brute numerical codes run faster and faster, these semi-analytical treatments still allow one to get a feel for what happens when one varies certain parameters (such as $M$).

It is useful to define 2 new variables $u$ and $v$:

$$u = \frac{d \ln \left( M(r) \right)}{d \ln r} = \frac{3 \rho(r)}{\bar{\rho}(r)} = \frac{\xi \theta^n}{(d \xi)}$$

$$v = -\frac{d \ln P(r)}{d \ln r} \left( \frac{1}{n+1} \right) = -\frac{3}{2} \frac{GM(r)}{r} \left[ \frac{3}{2} \left( \frac{kT}{\mu m_H} \right) \right] = -\frac{\xi}{\theta} \left( \frac{d \xi}{d \theta} \right)$$

so

$$\frac{u \ dv}{v \ du} = -\frac{u + v - 1}{u + nv - 3}.$$

The boundary conditions require $u = 3$, $v = 0$ \( \left( i.e., \frac{3\rho}{\bar{\rho}} \to 3 \text{ and } \frac{dP}{dr} \to 0 \text{ as } r \to 0 \right) \).

Solutions:

$$u = 3 - \frac{n \xi^2}{5} + \frac{19n^2 - 25n}{1050} \xi^4 - \frac{(472n^3 - 1275n^2 + 875n)}{283500} \xi^6 + \cdots$$

$$v = \frac{\xi^2}{3} \left[ 1 - \frac{(3n - 5)}{30} \xi^2 + \frac{(12n^2 - 39n + 35)}{1260} \xi^4 - \cdots \right]$$

\[\text{Figure 2.1} \] Solution for two common polytropes with physical interpretations. The solid lines represent the $E$ solutions which satisfy hydrostatic equilibrium at the origin. The dashed and dotted lines depict samples of the $F$ and $M$ solutions, respectively. The polytrope with $n = 1.5$ represents the solution for a star in convective equilibrium, while the $n = 3$ polytrope solution is what is expected for a star dominated by radiation pressure.
Isothermal Gas Spheres

Here $\Gamma = 1 \Rightarrow n = \infty$ and we cannot use the Lane-Emden equation.

Let $P = \frac{\rho k T}{\mu m_n} + \frac{1}{3} a T^4 = K \rho + D$ for $T = \text{const.}$ (i.e. isothermal)

So, $\frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi r^2 G \rho = \frac{d}{dr} \left( \frac{r^2}{\rho} K \frac{d\rho}{dr} \right)$. Let us make a change of variables:

$\rho = \rho_c e^{-\psi}$ and $r = \left[ \frac{K}{4\pi G \rho_c} \right]^{\frac{1}{2}} \xi = \alpha \xi$. With this we get $\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\psi}{d\xi} \right) = e^{-\psi}$.

Boundary conditions:

;r \rightarrow 0 \Rightarrow \xi \rightarrow 0 \text{ with } \rho \rightarrow \rho_c \text{ and } \psi \rightarrow 0 \text{ (so } e^{-\psi} \rightarrow 1)\n
;r \rightarrow 0 \Rightarrow \frac{dP}{dr} \rightarrow 0 \Rightarrow \frac{d\psi}{dr} = 0 \text{ at } \xi = 0

There are no analytic solutions!

$\psi = \frac{\xi^2}{6} - \frac{\xi^4}{120} + \frac{\xi^6}{1890} \cdots \text{ as } \xi \rightarrow 0$

For our physical variables, we get

$M(\xi) = 4\pi \left[ \frac{K}{4\pi G} \right]^{\frac{1}{2}} \frac{1}{\rho_c^{\frac{1}{2}}} \xi^2 \frac{d\psi}{d\xi}$

$\bar{\rho} = \frac{3 \rho_c}{\xi} \frac{d\psi}{d\xi}$

$P = K \rho + D$

$T = K \frac{\mu m_n}{k} = \text{const.}$

$r = \left[ \frac{K}{4\pi G \rho_c} \right]^{\frac{1}{2}} \xi$
In terms of the uv-plane,

\[ u = 3 - \frac{\xi^2}{5} + \frac{19}{1050} \xi^4 - \frac{118}{70875} \xi^6 + \cdots \]

\[ v = \frac{\xi^2}{3} \left[ 1 - \frac{\xi^2}{5} + \frac{\xi^4}{10} - \frac{\xi^6}{105} - \cdots \right] \]

**Figure 2.2** Solution for isothermal sphere in the uv plane. The solution is unique.

In the real universe, stars are usually not described by a unique solution throughout their entire volumes. Some stars have an isothermal core and a convective envelope. The latter can be described by a polytrope with n=1.5, but it does not reach the core. These other solutions are described in Chandra’s *Stellar Structure* in some detail.

In such a case, one integrates outward in the isothermal core until the envelope is reached. Pick the appropriate n for the envelope, then jump from the (u,v) of the core to the (u,v) of the envelope. For example, because P and T are continuous across the boundary, \( \frac{\rho_c}{\rho_e} = \frac{\mu_c}{\mu_e} \). Knowing \( \frac{\mu_c}{\mu_e} \) determines \( \frac{\rho_c}{\rho_e} \), etc.

Then follow the second (polytropic) solution outward.
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Closing Thoughts

Okay, we went through a lot of what seemed to be rather esoteric mathematical juggling to get these results. Why do all these mental gymnastics when the realistic detailed structure of a star can be calculated on a computer for any arbitrary equation of state (or states)? Not just because this was the way it was done in the age before computers. If you were a person writing a stellar structure code, how would you know if it was giving correct results unless you tested your results by comparing them to the analytic solutions in those cases where the latter is known?

Figure 2.3 A model star composed of two polytropes. The outer convective hydrogen envelope can be represented by a polytrope of index $n = 1.5$, while the helium core is isothermal. The discontinuous change in $u$ and $v$ resulting from the change in chemical composition can be seen as a jump from the isothermal core solution toward the origin and the appropriate $M$ solution for the envelope. Such a model can be expected to qualitatively represent the evolved phase of a red giant.