Lecture 10: Energy Transport

In the stellar atmosphere problem, we only considered one kind of energy transport -- *radiative transport*. In the stellar interior problem there are also times when energy is transported by *convection* or *conduction*. The latter is only important at very high densities, when material is degenerate, and we will postpone discussion of that until the lecture on white dwarf stars.

Radiative Energy Transport in stellar interiors:

This is the same as in the atmospheres problem except we only care about the flux, not the intensity, we do not care about the frequency dependence, and the geometry is spherical, not plane parallel.

We can use the gray body equations if we identify the frequency independent (gray) opacity as some “average”. The appropriate average to use is one that preserves the flux and is known as the *Rosseland mean opacity*. It is a weighted mean in which highest weight is given to frequencies at which the most energy is being transported. Since flux is highest where opacity is lowest, the average is calculated as an inverse.

\[
\frac{1}{\kappa} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} \, d\nu}{\int_0^\infty \frac{dB_\nu}{dT} \, d\nu} = \frac{\pi}{4\sigma T^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} \, d\nu
\]
Radiative Transfer in Spherical Coordinates:

\[ I_\nu(r, \theta) \]

\[ I_\nu(r + dr, \theta + d\theta) \]

From the geometry:

\[ dr = dl \cos \theta \]

\[ \cos \theta \frac{dI}{dr} = j - kI \]

\[ \frac{dI_r}{dr} = \frac{\partial I_\nu}{\partial r} + \frac{\partial I_\nu}{\partial \theta} \frac{d\theta}{dr} \]

From the geometry:

\[ \frac{d\theta}{dr} = \frac{d\theta}{dl} \cdot \frac{dl}{dr} = \left( -\frac{\sin \theta}{r} \right) \left( \frac{1}{\cos \theta} \right) \]

So, the equation of radiative transfer in spherical coords is:

\[ \cos \theta \frac{\partial I_\nu}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I_\nu}{\partial \theta} = j - kI \]
What saves some complications in the stellar interior is, however, the very small mean free path for a photon, of order a cm! Hence, we can assume that radiation field is isotropic except for a very tiny gradient. The photons “diffuse” out of the interior. In this case, we can use the same equations as for the plane parallel case, gray case, identifying $k$ now as the Rosseland mean opacity. Taking the moments of the equation of radiative transfer, as we did in Lecture 5, we find:

$$c \frac{dP_{rad}}{d\tau} = F$$

But

$$P_{rad} = \frac{1}{3} a T^4$$

where

$$a = \frac{4\sigma}{c}$$

$a$ is known as the “radiation constant”.

So

$$F = \frac{1}{3} ac \frac{dT^4}{d\tau} = \frac{4}{3} ac T^3 \frac{dT}{d\tau} = -\frac{4acT^3}{3k} \frac{dT}{dr}$$

In the interior problem, the more common variable is $L_r$ (the luminosity through a shell of radius $r$): $L_r \equiv 4\pi r^2 F$

Hence,

$$L_r = -\frac{16\pi r^2 acT^3}{3\kappa \rho} \frac{dT}{dr}$$
This equation dictates what the temperature gradient must be in the case of radiative transport in order to “push through” the amount of flux that is being provided by the star through either its nuclear energy generation or its gravitational contraction heating. Along with the equation of hydrostatic equilibrium and the mass conservation equation, it is one of the fundamental equations of stellar structure, but it only applies if the energy transport is by radiation.

Convective Energy Transport

When the rate of heating is too high, or when radiative transport is too inefficient due to a high opacity, a more efficient means of energy transport takes over -- convection. This involves actual motion of parcels of gas in “currents” or “winds” or “convective streams”. A precise description of how the gas moves is impossible -- it is too complex. We use only a simple, approximate model to describe the process, which is called the “mixing length” theory. The idea is that energy is put into a “parcel” of gas at a lower level in the star and, under the condition of convective instability (to be derived below), the parcel will rise for some “mixing length” (usually stated in terms of the number of pressure scale heights), before “dumping” its excess energy load. The gas in a convective zone is in “adiabatic” equilibrium, then -- it is not exchanging heat with its surroundings.
Adiabatic Relations in an ideal gas:

Recall from Thermodynamics that: \(dQ = dU + PdV\) where \(Q\) is the heat input per gm (or kg), \(U\) is the internal energy per gm, \(P\) is the pressure and \(V\) is the “specific volume” (i.e. volume of 1 gm of matter). This just states that if you add energy to a gas it will either go into internal energy or expansion of volume.

The response of a gas to changing its temperature is characterized by its “specific heat” (C) which can be measured at constant pressure or constant volume:

\[
C_P \equiv \left[ \frac{dQ}{dT} \right]_P = \left[ \frac{dU}{dT} \right]_P + \left[ P \frac{dV}{dT} \right]_P
\]

\[
C_V \equiv \left[ \frac{dQ}{dT} \right]_V = \left[ \frac{dU}{dT} \right]_V
\]

For an ideal gas

\[P = \frac{\rho}{\mu m_H} kT \text{ or } PV = \frac{kT}{\mu m_H}\]

Each particle has \(3/2\) kT of energy, so

\[U = \frac{1}{\mu m_H} \left( \frac{3}{2} kT \right) = \frac{3}{2} \frac{P}{\rho} = \frac{3}{2} PV\]
\[ C_P = \frac{1}{\mu m_H} \left( \frac{3}{2}k \right) + P \left[ \frac{1}{\rho} \frac{k}{\mu m_H} \right] = \frac{5}{2} \frac{k}{\mu m_H} \]

\[ C_V = \frac{3}{2} \frac{k}{\mu m_H} \]

\[ \gamma \equiv \frac{C_P}{C_V} = \frac{5}{3} \text{ (for an ideal gas)} \]

An adiabatic change is one in which \( dQ = 0 \). Eliminating \( V \) from the equations above, in this case leads to:

\[ \frac{d\ln P}{d\ln T} = \frac{C_P}{C_P - C_V} = \frac{\gamma}{\gamma - 1} \]

or, an expression for the “adiabatic temperature gradient”:

\[ \left( \frac{dT}{dr} \right)_{ad} = \left( 1 - \frac{1}{\gamma} \right) \frac{T}{P} \frac{dP}{dr} \]

This is the temperature gradient that is assumed to apply in convective regions. If the radiative temp gradient is steeper than this, then convection breaks out.
Note that since $T \propto P/\rho$ for an ideal gas, we can alternatively eliminate $T$ from the adiabatic relations and find $P \propto \rho^{\gamma}$

This, of course, means that a convective zone has a pressure - density structure that is given by a polytrope, of index

$$n = \frac{1}{\gamma - 1} = \frac{3}{2} \text{ for an ideal gas.}$$

Convection is important in two regimes. For higher mass stars, which use the CNO cycle, there is a strong $T$ dependence of the nuclear reactions, which leads to a high $T$ gradient, which encourages convection to break out in their cores. This is important for mixing the products of the nuclear reactions (helium, at first) over a larger volume than would otherwise be true and therefore prolonging the lifetime of the star on the main sequence.

For lower mass stars (a couple solar mass or less) the lower $T$ of their atmospheres leads to high opacity and inefficient radiative transport. They develop convective envelopes. The existence of convection in the solar envelope is seen in the changing patterns of solar “granulation”.
A typical granule is about 1000 km across and lasts about 20 minutes.
Notes on convection:

1. Strong magnetic fields on the surfaces of lower mass stars can inhibit convection in certain areas and cause cool regions to appear as highly magnetized and much cooler “spots”. These sun spots (or star spots) can persist for months or even years on some stars.

2. Pre-main sequence stars are fully convective at first and are, therefore, well modeled as polytropes of index 3/2. Also, the convection effectively mixes the star completely. Since lithium can undergo nuclear reactions at temperatures much lower than hydrogen, it is also depleted from the atmospheres of stars as time goes on. If we see a star with a strong lithium line it is a sign that it is still very young (only a few million years old at most).

3. Lack of a good theory of convection is one of the weakest links in our theory of stellar structure. The “mixing length”, usually taken to be 1 or 2 pressure scale heights, is a “free parameter” in stellar models that is adjusted so that the models fit the observations. This freedom weakens the power of the models to predict. Some corrections for the simplicity of the mixing length theory are employed in models, such as “convective overshoot” regions, where mixing goes beyond the actual region of convective instability, and “super adiabatic temperature gradients” where the T gradient is locally higher than adiabatic.
4. In an “ionization zone”, i.e. where hydrogen (or helium) is undergoing a transition from mostly neutral to mostly ionized, a good deal of energy can be put into a gas, without raising its temperature. This means that the specific heats can become very large and their ratio (gamma) is no longer 5/3 but closer to 1. This, in turn, means that the adiabatic temperature gradient becomes very small and the region becomes naturally convectively unstable.