Lecture 3: Specific Intensity, Flux and Optical Depth

We begin a more detailed look at stellar atmospheres by defining the fundamental variable, which is called the **Specific Intensity** or sometimes just the intensity, for short. It may be specified as a function of frequency, $I_\nu$, or of wavelength, $I_\lambda$. It is basically the same as flux except limited to those photons headed in a particular direction (i.e. confined within a certain solid angle).
Specific intensity is a nearly complete description of the radiant energy “field”. (The only thing not included is information about polarization.) Otherwise, I describes how many photons of what energies are present and what directions they are headed. Everything else of interest (e.g. the flux through any particular area oriented any particular way, the radiation pressure, the radiant energy density, etc. can be derived from the intensity (notice I used its nickname here!)

Let’s get to know I by noting some of its properties. Since,

\[ \int_0^\infty I_\lambda d\lambda = \int_0^\infty I_\nu d\nu \]

to switch from frequency to wavelength use the following relations:

\[ I_\lambda = -I_\nu \frac{d\nu}{d\lambda} = I_\nu \frac{c}{\lambda^2} = I_\nu \frac{\nu^2}{c} \]

The units of \( I_\nu \) are (ergs per sec per cm^2 per steradian per Hz).

Note that \( I_\nu \) is independent of distance. It is like a surface brightness ... ie. magnitudes per square arc-sec. Surface brightness is not dependent on distance (neglecting cosmological effects!). Stars don’t “fade” with distance, they get smaller.....more later.
A very important quantity that can be derived from specific intensity is the flux through any surface. All we have to do is add up all the energy going through the surface, regardless of direction. Remember that what we mean by “flux” might be thought of as the “net flux” -- i.e. the difference between flux going one direction (normal to the chosen area) and going the opposite way (anti-normal). This follows immediately when we consider that the flux must be a weighted sum over the intensity, since photons parallel to our chosen area do not actually go through it and contribute zero to the flux. Some reflection (perhaps a decade will do) will convince you that the intensity needs to be weighted by the cosine of the angle between the normal to the area chosen and the direction of the rays. For now, accept it as the wisdom of those who have gone before you!

\[ F_\nu = \int I_\nu \cos \theta d\Omega = \int \int I_\nu \cos \theta \sin \theta d\theta d\phi \]

As a trivial example, we show that the flux in an isotropic radiation field is zero!

\[ F_\nu = \int_{4\pi} I_\nu \cos \theta d\Omega = I_\nu \int_0^{2\pi} \int_0^{\pi} \cos \theta \sin \theta d\theta d\phi = 2\pi I_\nu \int_0^{\pi} \cos \theta \sin \theta d\theta = 0. \]
Note on doing these types of integrals, which appear a lot in astrophysics. Use substitution \( u = \cos(\theta) \) from which \( du = -\sin(\theta) \) follows.

\[
\int_{0}^{\pi} \cos\theta \sin\theta d\theta = \int_{1}^{-1} -x dx = \left[ \frac{x^2}{2} \right]_{1}^{-1} = 0
\]

As a less trivial and very important example, let's calculate the flux at the surface of a star, assuming that all of the intensity is outward directed (i.e. space is black!) and the outward bound intensity is independent of direction, i.e. \( I_{\nu} \neq I_{\nu}(\theta, \phi) \)

\[
F_{\nu} = \int_{4\pi} I_{\nu} \cos\theta d\Omega = 2\pi I_{\nu} \int_{0}^{\pi/2} \cos\theta \sin\theta d\theta = 2\pi I_{\nu} \int_{1}^{0} -x dx = \pi I_{\nu}
\]

The famous equation \( F = \sigma T^4 \) follows by integrating the Planck function over all frequencies (see problem set)!

The inverse square law follows for stars as follows: Since Intensity is independent of distance, and assuming that stars are very far away and nearly point sources, one can take \( \cos\theta = 1 \) and \( F_{\nu} = I_{\nu} \Omega \)

\[
\Omega = \frac{\pi R^2}{d^2} \quad \text{so} \quad F \propto d^{-2}
\]

So, again note that stars get fainter with distance because they subtend a smaller solid angle, not because they “fade” in any way.
In Section 9.1 of the text, the authors show you how to derive other quantities of interest from intensity. We don’t need to repeat all the stuff here, but will highlight a few items:

**Mean intensity:**

\[ J_\nu \equiv \langle I_\nu \rangle = \frac{1}{4\pi} \int_{4\pi} I_\nu d\Omega = \frac{1}{2} \int_0^\pi I_\nu(\theta) \sin \theta d\theta \]

**Energy Density:**

\[ u_\nu = \frac{1}{c} \int_{4\pi} I_\nu d\Omega = \frac{2\pi}{c} \int_0^\pi I_\nu(\theta) \sin \theta d\theta = \frac{4\pi}{c} J_\nu \]

Note: energy density is the ergs per cubic cm (or joules per cubic m) in the radiation field. The factor of c arises when one considers the volume in which light is “confined”. In time dt the light travels a distance ds=cdt. See book for more on this derivation. This quantity is sometimes called monochromatic energy density. To get the full radiant energy density (at all frequencies) integrate! In a blackbody in STE, \( I = J = B \) (the Planck function). In this case, we can calculate the energy density as:

\[ u = \int_0^\infty u_\nu d\nu = \frac{4\pi}{c} \int_0^\infty B_\nu(T) d\nu = \frac{4\sigma T^4}{c} \equiv aT^4 \]

The new constant a is called the “radiation constant”
Radiation Pressure (see book for derivation): The radiation pressure can be considered the second “moment” of intensity, in the sense of a moment arm. The first moment is flux. The effect of photons on the net flux depends on their direction, so we weight photons according to the angle they make with the normal to the area considered. In the radiation pressure calculation, as we go to higher inclinations, not only are there fewer photons making it through dA but each photon carries a diminished contribution to the normal force (i.e. the pressure). Again, the factor is \( \cos(\theta) \), so radiation pressure is the second moment of intensity.

\[
P = \int_{0}^{\infty} \frac{1}{c} \int_{4\pi} I_\nu \cos^2(\theta) d\Omega d\nu
\]

For an isotropic radiation field:

\[
P = \int_{0}^{\infty} \frac{4\pi}{3c} J_\nu d\nu
\]

And, for a black body in STE, where \( J = B \):

\[
P = \int_{0}^{\infty} \frac{4\pi}{3c} B_\nu d\nu = \frac{4\sigma T^4}{3c} = \frac{1}{3} aT^4 = \frac{1}{3} u
\]
If the Universe were empty then the problem of radiative transfer would be easy and the solution of the equation would be \( I_\nu = I_\nu(0) \). That is, intensity at the location of the observer would be the same as intensity at the location of the source. Intensity would be constant.

However, the Universe is full of matter that will both absorb energy from the beam of light and add energy to it. A description of how the intensity changes as it moves through space (with matter in it) is given by the equation of radiative transfer. This may be thought of as the fundamental equation of stellar atmospheres and is one of the fundamental equations of all of astrophysics.

Derivation is quite simple, because only two things can happen to light as it passes through a small volume of space -- the intensity is either reduced by absorption or scattering of radiation out of the beam, or it is increased by the emission of photons by matter in the volume or scattering of photons originally headed in other directions into the same direction as the beam is headed. Two parameters are needed to describe these two possibilities: an absorption coefficient (k) and an emission coefficient (j). Both, of course, are functions of frequency (or wavelength).
If we consider a small volume \( dV = dA \, ds \) where \( ds \) is along the beam direction and \( dA \) is perpendicular to it, then the amount of energy added to the beam in time \( dt \) and in frequency interval \( d(\nu) \) and confined to within \( d(\Omega) \) of the beam direction in going from \( s \) to \( s+ds \) is given by

\[
\text{Radiant energy added to beam} = j_\nu dA dA d\nu d\Omega
\]

Likewise, the amount of radiant energy scattered from the beam or removed by absorption, is given by

\[
\text{Radiant energy removed from beam} = k_\nu I_\nu dA dA d\nu d\Omega
\]

Note that the amount of energy removed depends on how strong the beam is going in....if no energy comes in then none gets removed! On the other hand, the amount added to the beam is independent of incoming intensity. The only exception to this is laser emission (which sometimes happens in space!) Usually this effect is treated as “negative absorption” because it is emission which does depend on the level of the incoming radiation. We will not discuss laser emission any further in this course.

The change in intensity is equal to the amount added - the amount removed:

\[
[I_\nu(s + ds) - I_\nu(s)] dA d\nu d\Omega = [j_\nu - k_\nu I_\nu] dA dA d\nu d\Omega
\]
Simplifying, and taking the limit as $ds$ goes to zero, we have:

$$
\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu
$$

which is one form of the Equation of Radiative Transfer. A more common and useful form comes by dividing both sides of the equation by $k$ and making two definitions, yielding:

$$
\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu
$$

where $S_\nu \equiv \frac{j_\nu}{k_\nu}$ is called the “Source Function”

and $d\tau_\nu \equiv k_\nu ds$ is called the “optical depth”

Some examples may help clarify these definitions and their point. Consider a gas in STE, then there is no change in intensity with position so the left hand side of the equation is zero and $S_\nu = I_\nu = B_\nu$

The source function is also equal to the Planck function, at least approximately, in other situations, such as Local Thermodynamic Equilibrium (LTE) where the local kinetic temperature, $T$, is used for the assumption that $S_\nu = B_\nu(T)$
To get a feel for the meaning of optical depth, consider the transfer of radiation through a medium where the absorption and scattering (together often referred to as “extinction”) is important, but the emission is not. An example of this is the propagation of star light through the interstellar medium. At optical wavelengths the ISM adds nothing of importance to the beam because the matter is so cold. However, the dust in space does cause extinction if the path length is long enough. In this case, the source function is zero, so the transfer equation can be written as:

\[
\frac{dI_\nu}{d\tau_\nu} = -I_\nu \quad \text{or} \quad \frac{dI_\nu}{I_\nu} = -d\tau_\nu
\]

This equation is easily solved by integrating both sides. It requires one boundary condition, which we take to be \( I = I(0) \) at \( \tau = 0 \). Then, at any other optical depth (\( \tau \)) we have

\[
I_\nu = I_\nu(0)e^{-\tau_\nu}
\]

This tells us that when we view a star through an optical depth of 1, its intensity is reduced by \( 1/e \) compared to what it would be if there were no extinction. So, we can think of optical depth as the number of e-folds that the light is attenuated. If \( \tau \ll 1 \) the medium is called “optically thin” and there is little absorption. If \( \tau \gg 1 \) the medium is called “optically thick” and we can’t see through it! It is opaque.
One can also relate the optical depth to the reduction of brightness (in magnitudes) suffered by the star due to the extinction. Using the magnitude equation we find the difference in magnitude between the extincted and unextincted star as:

\[ \Delta m = -2.5 \log_{10} \frac{I_\nu}{I_\nu(0)} = -2.5 \log_{10} e^{-\tau_\nu} = 1.086 \tau_\nu \]

So, roughly speaking, the optical depth is about equal to the number of magnitudes of extinction suffered by the beam.

If we think about it on a microscopic level, the attenuation (extinction) of light is due to its interaction with bits of matter -- atoms, electrons, molecules or larger stuff, collectively called “dust” or “grains”. While there can be many different kinds of particles present, let's consider for simplicity that there is only one kind, that there are \( n \) such particles per unit volume and that each particle has a “cross-section” for extinction equal to \( \sigma \). Then we may write that

\[ d\tau_\nu = k_\nu ds = n\sigma_\nu ds \]

If we integrate over a path length such that \( \tau = 1 \) and call that path length “s” then we can identify \( s \) as the “mean free path” of a photon and the optical depth (\( \tau \)) as the number of mean free paths to the source of the beam.
Clearly,

\[ s \text{ (mean free path)} = \frac{1}{n\sigma_{\nu}} \]

Put another way, the mean free path is the distance through the medium that one has to go until the cross-sections of all the absorbing particles would block the entire beam (if they did not overlap each other). Because they DO overlap each other, some light does get through even when the path length through the medium is 1 mean free path. How much light? Well, the intensity is reduced by \(1/e = 0.368\), so only about 63% of the light is actually blocked. 37% makes it through!