Lecture 4: Radiative Transfer, Kirchhoff’s Laws

The Equation of Radiative Transfer, derived in the last lecture, may be written as:

\[
\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu
\]

We have already seen one solution, for the special case that \( S = 0 \), in which case:

\[
I_\nu = I_\nu(0)e^{-\tau_\nu}
\]

More generally, this is a first order differential equation with constant coefficients and it has a general solution which involves one boundary condition. The general solution is:

\[
I_\nu = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)e^{-(\tau_\nu-\tau'_\nu)}d\tau'_\nu
\]

Observer

\[
\tau_\nu - \tau'_\nu
\]

\[
S_\nu(\tau'_\nu)
\]

\[
\tau'_\nu
\]

\[
0
\]

Source
Consideration of various cases will help make sense of this:

1) If no matter is present between us and source then the independence of intensity on distance is recovered. \((I = I(0))\).

2) If only extinction is important, then the exponential decay with number of mean free paths (i.e. optical depth) is also recovered.

3) In the case that \(S\) is independent of tau over some region, the equation reduces to:

\[
I_{\nu} = I_{\nu}(0)e^{-\tau_{\nu}} + S_{\nu} \int_{0}^{\tau_{\nu}} e^{-(\tau_{\nu}-\tau'_{\nu})} d\tau'_{\nu} = I_{\nu}(0)e^{-\tau_{\nu}} + S_{\nu}(1 - e^{-\tau_{\nu}})
\]

For the case of no background source and small optical depth, this further reduces to:

\[
I_{\nu} = \tau_{\nu}S_{\nu}
\]

The equation of radiative transfer and its solutions make it possible to understand Kirchoff’s laws of spectroscopy in a new light (ha ha):
1) **A hot solid or dense gas produces a continuous spectrum:**
   In the optically thick case $I = S = B$ so we see the Planck curve.

2) **A hot thin gas produces an emission spectrum:**
   In an optically thin gas $I = \tau S$. Where $\tau$ is large, $I$ is large. $\tau$ is large only in the region of spectral lines. So, one sees a very faint continuum (often invisible) with emission lines superimposed.
3) **A cool, thin gas seen in front of a hot source produces absorption lines:**

   In the continuum region, tau is low since the gas is optically thin and we see primarily the background source. At wavelengths of spectral lines, tau is large and we see the intensity characteristic of the temperature of the cool gas. Since this is lower, these appear as absorption features against the background source. If the gas is MUCH lower in temperature then the intensity in the lines goes to zero. If it is only a little lower, then the intensity does not go to zero. (The solar spectrum can be used to illustrate this using lines in the Earth’s atmosphere and lines in the Sun’s atmosphere.)

The detailed calculation of $k_{\text{nu}}$, the opacity, is a very tough and extended problem within astrophysics. Calculating cross-sections as a function of frequency for all sorts of atoms and molecules and even dust grains is a MESS. We will touch on this “can of worms” later but first let’s see what progress can be made without worrying about opacity. Historically, this is what people did and the progress they made in the early part of the last century in understanding stellar atmospheres is quite impressive. This is embodied in the Gray Atmosphere model, work which was largely carried on by Sir Arthur Eddington, the famous British astrophysicist.