1. *Snowboards.*

(a) Both the equilibrium quantity and price will fall as a result of the shift in demand.

\[ a - p = 3p \]
\[ p^* = \frac{a}{4} \]
\[ \frac{dp^*}{da} = \frac{1}{4} > 0 \]

(b) There is no deadweight loss because there is no potential consumer or producer surplus that is not achieved at the market equilibrium. There is a reduction in the total consumer and producer surplus obtained from this market, but it is solely due to a change in consumer tastes, not to a market failure or distortion of the market.

(c) First set demand equal to supply, then solve for \( p^* \), and then differentiate:

(d) Again set demand equal to supply. Then totally differentiate and try to rearrange into elasticity form:

\[ x(p, a) = s(p) \]
\[ \frac{\partial x}{\partial p} dp + \frac{\partial x}{\partial a} da = \frac{\partial s}{\partial p} dp \]
\[ dp \left( \frac{\partial x}{\partial p} - \frac{\partial s}{\partial p} \right) = -\frac{\partial x}{\partial a} da \]
\[ \frac{dp}{da} = -\frac{\frac{\partial x}{\partial a} p}{\frac{\partial x}{\partial p} - \frac{\partial s}{\partial p}} \]
\[ \frac{dp}{da} = -\left( \frac{\frac{\partial x}{\partial a} p}{\frac{\partial x}{\partial p} - \frac{\partial s}{\partial p}} \right) \]

(e) That’s the best I can do for part (d). But if we define \( \epsilon_a = \frac{\partial x}{\partial a} \) as the elasticity of demand with respect to shifter \( a \), and if we look for the elasticity of \( p \) with respect to \( a \) instead of just the derivative, we can do a little better. Continuing from the end of part (d):

\[ \frac{dp}{da} \frac{a}{p} = -\frac{\frac{\partial x}{\partial a} \frac{a}{p}}{\epsilon_D - \epsilon_S} \]
\[ \epsilon_{pa} = \frac{\frac{\partial x}{\partial a} \frac{a}{p}}{\epsilon_D - \epsilon_S} \]
\[ \epsilon_{pa} = \frac{\epsilon_a}{\epsilon_D - \epsilon_S} \]

2. EducatedMothers_a.

(a) Yes, there is a positive externality. The mother chooses the amount of education, but her children benefit directly and all other children benefit to the extent that the average level of education rises.

The externality may or may not be internalized depending on how you interpret the problem. The hard-nosed economist would answer that any concern the woman has for her child should already be reflected in her utility function. After all, her utility function is supposed to be a complete measure of
her happiness – if she cares about her child, then that should be in there.

In any case, it seems very unlikely that the mother would take account of the small effect that her own education has on the average level of education, so a positive externality should persist. Intuitively, the mother takes account of her own utility only, while a social planner would also take account of the effect on the average level. Thus, the socially optimal e would be higher than the free-market e.

(b) Using the hard-nosed logic from (a), the woman’s MRS is:

\[ MRS = -\frac{MU_x}{MU_e} = -\frac{\frac{3}{4}x^{-1/4}e^{1/4}}{\frac{1}{4}x^{3/4}e^{-3/4}} = -\frac{3e}{x} \]

To find her demand curve, we set the MRS equal to the ratio of the prices and then substitute into the demand function:

\[ -\frac{3e}{x} = -\frac{1}{p_e} \Rightarrow 3p_e e = x \]

\[ p_e e + x = 2000 \Rightarrow 4p_e e = 2000 \Rightarrow e(p_e) = \frac{2000}{4p_e} \]

(c) The social planner’s utility includes the 1000 mothers and their 3000 children:

\[ u_s(e, x) = 1000x^{3/4}e^{1/4} + 3000 \cdot 12e^{1/16}e^{3/16} \]

Note the way the big E is just replaced by little e for the social planner since the planner will decide every woman’s e and thus determine the average at the same time. The social planner’s MRS is:

\[ MRS = -\frac{MU_x}{MU_e} = -\frac{1000\frac{3}{4}x^{-1/4}e^{1/4}}{1000\frac{1}{4}x^{3/4}e^{-3/4} + 36000\frac{1}{4}e^{-3/4}} \]
Since there is more “stuff” in the denominator for the social planner, the social planner will always have a smaller (in absolute value) MRS than the mother on her own.

Now set the social MRS equal to the slope of the budget line \((-p_x/p_e = -1/p_e)\):

\[
\begin{align*}
p_e 1000 \frac{3}{4} x^{-1/4} e^{1/4} &= 1000 \frac{1}{4} x^{3/4} e^{-3/4} + 36,000 \frac{1}{4} e^{-3/4} \\
p_e &= \frac{1}{3} e + 12 \frac{x^{0.25}}{e} \\
p_e e &= \frac{1}{3} x + 12 x^{0.25}
\end{align*}
\]

And substitute into the budget line:

\[
p_e e + x = 2000 \Rightarrow \frac{1}{3} x + 12 x^{0.25} + x = 2000 \Rightarrow \frac{4}{3} x + 12 x^{0.25} = 2000
\]

This is slightly tricky to solve. Let \(y = x^{0.25}\). Then we can rewrite the equation as:

\[
\frac{4}{3} y^4 + 12 y = 2000
\]

Using the polynomial route finder on a calculator, the only positive real route is \(y = 6.16\), which implies that \(x = 1445\). Then using the formula for \(p_e e\), we can find the social demand curve for education:

\[
p_e e = \frac{1}{3} 1445 + 12(1445^{0.25}) \Rightarrow e^{soc}(p_e) = \frac{555.6}{p_e}
\]

This is similar to the private demand curve, but with a larger numerator. It is clear that the social demand curve will lie above and to the right of the private demand curve. The Pigouvian subsidy that shifts the private demand curve up to match the social demand curve solves:

\[
\frac{500}{(1 - s)p_e} = \frac{555.6}{p_e} \Rightarrow s = 10\%
\]

(b) The payout is subsidy times quantity. At equilibrium, the quantity will be \( x(p^*) = s(p^*) \), so we need to find \( p^* \):

\[
x(p) = s(p) \Rightarrow a - bp + bg = \alpha p \Rightarrow p = \frac{a + bg}{\alpha + b}
\]

Then the equilibrium quantity is

\[
s(p^*) = \alpha \frac{a + bg}{\alpha + b}
\]

The government payout is

\[
R = g s(p^*) = g \alpha \frac{a + bg}{\alpha + b}
\]

It changes with \( g \) according to

\[
\frac{dR}{dg} = a \frac{a + bg}{\alpha + b} + g \alpha \frac{b}{\alpha + b}
\]

The first and second terms are both clearly positive since all parameters and \( g \) are positive. So as the government raises the subsidy, it will definitely pay out more money.
(c) Elasticity is \( \frac{dR}{dg} \). Note that we could rewrite the answer to part (b) as

\[
\frac{dR}{dg} = \frac{R}{g} + g \alpha \frac{b}{\alpha + b}
\]

This makes it easier to find that

\[
\frac{dR}{dg} = 1 + g^2 \alpha \frac{b}{\alpha + b} \left( g \alpha \frac{a + bg}{a + b} \right)^{-1}
\]

We can simplify a bit:

\[
\frac{dR}{dg} = 1 + g \frac{b}{\alpha + b} \frac{a + b}{a + bg} = 1 + g \frac{b}{a + bg}
\]