PROJECT 4: THE DOUBLE-DELTA MOVE AND S-EQUIVALENCE

1. Part I: Required

Two square matrices $V$ and $W$ are unimodular congruent if there is a matrix $P$ such that $W = P^T V P$ where $\det P = \pm 1$.

**Problem 1.** Give an example of two matrices that are unimodular congruent.

Given, a square matrix $V$, $W$ is a row enlargement of $V$ (and $V$ is a row reduction of $W$) if

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & u \\ 0 & v & V \end{pmatrix},$$

where $x$ is an integer, $u$ is a row vector and $v$ is a column vector.

**Problem 2.** Give an example of square matrices $V$ and $W$ where $W$ is a row enlargement of $V$.

Given, a square matrix $V$, $W$ is a column enlargement of $V$ (and $V$ is a column reduction of $W$) if

$$W = \begin{pmatrix} 0 & 1 & 0 \\ 0 & x & u \\ 0 & v & V \end{pmatrix},$$

where $x$ is an integer, $u$ is a row vector and $v$ is a column vector.

**Problem 3.** Give an example of square matrices $V$ and $W$ where $W$ is a column enlargement of $V$.

Two square matrices with integer entries are S-equivalent if they are related by a sequence of unimodular congruences, row enlargements, row reductions, column enlargements, and column reductions.

Two knots are S-equivalent if they have S-equivalent Seifert matrices.

**Problem 4.** Give an example of two S-equivalent knots. You should find their Seifert matrices and show that they are S-equivalent.

2. Part II: Extra Credit

In 2003, Swatee Naik and Ted Stanford showed that two knots are S-equivalent if and only if they are related by a sequence of double-delta moves. (See Figure 1.) Their paper was called “A move on diagrams that generates S-equivalence of knots” and appeared in the Journal of Knot Theory and its Ramifications.

![Figure 1. The double-delta move]
**Problem 5.** *Give a proof that if two knots are related by a sequence of double-delta moves, then they are S-equivalent. (In other words, prove the easier half of their theorem.)*

It also turns out that two knots are S-equivalent if and only if they have isomorphic Blanchfield linking forms (which I studied in my thesis). There is interest in understanding how Blanchfield linking forms are related to knot Floer homology which can be computed combinatorially from a grid diagram for the knot. Therefore one possible beginning of understanding how the Blanchfield linking form is related to knot Floer homology is to figure out how to define a grid version of the double-delta move.

**Problem 6.** *Define a grid version of the double-delta move.*