Hermitian Operators

1. Definition: The operator \( A \) is hermitian if

\[
\int_{-\infty}^{+\infty} \phi^* A \psi \, dx = \int_{-\infty}^{+\infty} (A \psi)^* \phi \, dx
\]

where \( \psi \) and \( \phi \) are well-behaved wavefunctions.

2. The operators \( x, p = -i \frac{\partial}{\partial x}, \) and \( H \) are all hermitian, although the proofs are sometimes not trivial. To prove that \( p \) is hermitian requires, for example, integration by parts.

3. Note that the right hand side of eq (1) can be written as \( \int_{-\infty}^{+\infty} (A\psi)^* \phi \, dx \), where it is understood that \( A \) operates only on \( \psi \). That is, hermitian operators can operate on bra’s to the left as well as on kets to the right.

4. If \( A \) is hermitian then \( \langle \psi | A | \phi \rangle = \langle \phi | A | \psi \rangle^\dagger \). If \( A \) is hermitian and the matrix element is real, then \( \langle \psi | A | \phi \rangle = \langle \phi | A | \psi \rangle \).

5. The symbol for hermitian conjugation is a superscript dagger, \( \dagger \), as in, the hermitian conjugate of \( A \) is \( A^\dagger \). \( A = A^\dagger \), means \( A \) is hermitian.

6. The hermitian conjugate of a matrix is a new matrix with elements the complex conjugate of the transposed (reflected across the diagonal) elements of the original matrix.

\[
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & j
\end{pmatrix}^\dagger = \begin{pmatrix}
a^* & d^* & g^* \\
b^* & e^* & h^* \\
c^* & f^* & j^*
\end{pmatrix}
\]

If a matrix is equal to its hermitian conjugate, then that matrix is said to be hermitian. The following is an hermitian matrix,

\[
\begin{pmatrix}
7 & 5i & 6 \\
-5i & 9 & i \\
6 & -i & 47
\end{pmatrix}
\]

7. If we take the hermitian conjugate of operators and wavefunctions, the order is reversed and bras become kets and kets become bras,

\[
\langle n|ABC|m\rangle^\dagger = \langle m|C^\dagger B^\dagger A^\dagger|n\rangle.
\]

8. Hermitian operators have real eigenvalues. This is why \( H \) must be hermitian; energies are real.

9. The Hermitian conjugate of a number is just its complex conjugate. The hermitian conjugate of a real matrix is the transpose of the matrix. Hermitian conjugation is the generalization of complex conjugation to the realm of operators and matrices.