4-1. Which of the following candidates for wave functions are normalizable over the indicated intervals?

a. \( e^{-x^2/2} \ (\infty, \infty) \)

b. \( e^{x} \ (0, \infty) \)

c. \( e^{i\theta} \ (0, 2\pi) \)

d. \( \sinh x \ (0, \infty) \)

e. \( xe^{-x} \ (0, \infty) \)

\[ \sqrt{a} \]

\[ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-x^2} dx \] normalizable since \( \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \)

\[ x \]

\[ \int_{0}^{\infty} e^x e^{-2x} dx = \int_{0}^{\infty} e^{-x} dx = \frac{1}{5} e^{-x} \bigg|_{0}^{\infty} = \infty \] not normalizable

\[ \sqrt{c} \]

\[ \int_{0}^{\frac{\pi}{2}} (e^{i\theta}) e^{i\phi} d\theta = \int_{0}^{\frac{\pi}{2}} e^{i\theta} d\theta = \int_{0}^{\frac{\pi}{2}} d\theta = \frac{\pi}{2} \] normalizable

\[ x \]

\[ \int_{0}^{\infty} (e^x - e^{-x}) \sinh x \ dx \]

\[ \int_{0}^{\infty} \left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^x + e^{-x}}{2} \right) dx = \int_{0}^{\infty} \left( e^{2x} + e^{-2x} \right) dx \]

\[ \sqrt{e} \]

\[ \int_{0}^{\infty} x^2 e^{-x} dx = \frac{2!}{3!} \] (integral table) normalizable.
In this problem, we will prove that the form of the Schrödinger equation imposes the condition that the first derivative of a wave function be continuous. The Schrödinger equation is

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E - V(x)]\psi(x) = 0$$

If we integrate both sides from $a - \epsilon$ to $a + \epsilon$, where $a$ is an arbitrary value of $x$ and $\epsilon$ is infinitesimally small, then we have

$$\frac{d\psi}{dx}\bigg|_{x=a+\epsilon} - \frac{d\psi}{dx}\bigg|_{x=a-\epsilon} = \frac{2m}{\hbar^2} \int_{a-\epsilon}^{a+\epsilon} [V(x) - E] \psi(x) dx$$

Now show that $d\psi/dx$ is continuous if $V(x)$ is continuous.

$$\psi(a+\epsilon) = \psi(a-\epsilon) = \psi(a)$$

If $V$ continuous then $V(a+\epsilon) = V(a-\epsilon) = V(a)$

$$= \frac{2m}{\hbar^2} \left[ V(a) - E \right] \psi(a) 2\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0$$

$$\frac{d\psi}{dx}\bigg|_{x=a+\epsilon} - \frac{d\psi}{dx}\bigg|_{x=a-\epsilon} = 0 \quad \text{or} \quad \frac{d\psi}{dx} \text{ is continuous}$$

Suppose now that $V(x)$ is not continuous at $x = a$, as in

![Graph of a piecewise-continuous potential function](image)

Show that

$$\frac{d\psi}{dx}\bigg|_{x=a+\epsilon} - \frac{d\psi}{dx}\bigg|_{x=a-\epsilon} = \frac{2m}{\hbar^2} [V_i + V_r - 2E] \psi(a) \epsilon$$

If $V$ is not continuous then RHS of (1)

$$= \frac{2m}{\hbar^2} \int_{a-\epsilon}^{a+\epsilon} [V(x) - E] \psi(x) dx$$

It is easy to think about the two halves separately

$$= \frac{2m}{\hbar^2} \int_{a-\epsilon}^{a} [V(x) - E] \psi(x) dx + \int_{a}^{a+\epsilon} [V(x) - E] \psi(x) dx$$

$$\frac{d\psi}{dx}\bigg|_{x=a+\epsilon} - \frac{d\psi}{dx}\bigg|_{x=a-\epsilon} = \frac{2m}{\hbar^2} \left[ (V_i - E) \psi(a) \epsilon + (V_r - E) \psi(a) \epsilon \right]$$

$$= \frac{2m}{\hbar^2} \left[ V_i + V_r - 2E \right] \psi(a) \epsilon$$

(Continued)
so that \( \frac{d\psi}{dx} \) is continuous even if \( V(x) \) has a finite discontinuity. What if \( V(x) \) has an infinite discontinuity, as in the problem of a particle in a box? Are the first derivatives of the wave functions continuous at the boundaries of the box?

If \( V(x) \) has an infinite discontinuity, then

\[
\frac{\psi_2}{\psi_1} \left[ V_2 + V_1 - \beta \text{E} \right] \psi_1(x) \xrightarrow{\text{as } \beta \to \infty} \text{a constant}
\]

and as \( \beta \to \infty \), the product can go to any number, not necessarily zero.

Thus \( \frac{d\psi}{dx} \bigg|_{a-} - \frac{d\psi}{dx} \bigg|_{a+} \) can be non-zero.

If \( \psi \) can be discontinuous at \( a \) can be spiked (continuously).

4-6. Calculate the values of \( \sigma_v^2 = \langle E^2 \rangle - \langle E \rangle^2 \) for a particle in a box in the state described by

\[
\psi(x) = \left( \frac{630}{a^3} \right)^{1/2} x^2 (a-x)^2, \quad 0 \leq x \leq a
\]

\[
\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}
\]

\[
\sigma_v^2 = \langle E^2 \rangle - \langle E \rangle^2 = \langle \psi | \hat{H}^2 | \psi \rangle - \langle \psi | \hat{H} | \psi \rangle^2
\]

\[
\sigma_v^1 = \frac{6}{a^3} \int_0^a x^2 (a-x)^2 \frac{d}{dx} \left[ x^2 (a-x)^2 \right] - \left( \frac{630}{a^3} \right)^{1/2} \int_0^a x^2 (a-x)^2 \left( \frac{x^2}{a^4} \right) \frac{d^2}{dx^2} \left[ x^2 (a-x)^2 \right]
\]

\[
\frac{d}{dx} \left[ x^2 (a-x)^2 \right] = \frac{d}{dx} \left[ (a^2 - 2ax + x^2) \right] = \frac{d}{dx} \left[ x^2 - 2ax + a^2 \right]
\]

\[
= 4x^3 - 6ax^2 + 2a^2 x
\]

\[
\frac{d^2}{dx^2} \left[ x^2 (a-x)^2 \right] = 12x^2 - 12ax + 2a^2
\]

\[
\frac{d^3}{dx^3} \left[ x^2 (a-x)^2 \right] = 24x - 12a
\]

\[
\frac{d^4}{dx^4} \left[ x^2 (a-x)^2 \right] = 24
\]
\[ \sigma_E^2 = \frac{630}{a^9} \int_0^a x^3 (a-x)^2 \frac{t^4}{y^m} \cdot 2y \, dx \]

\[ \left( \frac{630}{a^9} \int_0^a x^2 (a-x) \left( -\frac{t^4}{2m} \right) (12x^2 - 12ax + a^2) \, dx \right)^2 \]

\[ \sigma_E^2 = \frac{630}{a^9} 2y \frac{t^4}{y^m} \int_0^a \left( x^4 - 2ax^3 + a^2x^2 \right) \left( 12x^2 - 12ax + a^2 \right) \, dx \]

\[ \left\{ \frac{630}{a^9} \left( -\frac{t^4}{2m} \right) \int_0^a \left[ 6x^6 - 12ax^5 + 6a^2x^4 - 4a^3x^3 \right] \, dx \right\}^2 \]

\[ \sigma_E^2 = \frac{630}{a^9} 2y \frac{t^4}{y^m} \left( \frac{1}{5} - \frac{2}{y^m} \right) x^5 - \left\{ \frac{630}{a^9} \left( -\frac{t^4}{2m} \right) \left[ \frac{12}{7} - 6 + \frac{38}{5} - 4 + \frac{21}{3} \right] a^7 \right\} \]

\[ \sigma_E^2 = \frac{630}{a^9} \frac{t^4}{y^m} \left( \frac{1}{5} - \frac{1}{2} + \frac{1}{3} \right) a^5 - \left\{ \frac{630}{a^9} \left( -\frac{t^4}{2m} \right) \left[ \frac{12}{7} - 6 + \frac{38}{5} - 4 + \frac{21}{3} \right] a^7 \right\} \]

\[ \sigma_E^2 = 1.26 \frac{t^4}{a^9 m^2} - \left( \frac{630}{a^9} \frac{t^4}{y^m} - \frac{(a^2)}{4 (105)^2} \right) \]

\[ \sigma_E^2 = 1.26 \frac{t^4}{a^9 m^2} - \frac{36}{a^9 m^2} \]

\[ \sigma_E^2 = 90 \frac{t^4}{a^9 m^2} \]
Problem 4-6

In[1]:= \( \psi = \sqrt{\left\{(630 / a^9) \right\} x^2 (a - x)^2} \)

Out[1]= \( 3 \sqrt{70} \left( \frac{1}{a^9} \right) (a - x)^2 x^2 \)

In[2]:= AveESq = \( \int_0^a \psi \left( -\frac{h^2}{2 \text{ m}} \right)^2 * D[\psi, \{x, 4\}] \, dx \)

Out[2]= \( \frac{126 h^4}{a^4 \text{ m}^2} \)

In[3]:= SqAveE = \( \left( \int_0^a \psi \left( -\frac{h^2}{2 \text{ m}} \right) * D[\psi, \{x, 2\}] \, dx \right)^2 \)

Out[3]= \( \frac{36 h^4}{a^4 \text{ m}^2} \)

In[4]:= \( \sigma \text{SqE} = \text{AveESq} - \text{SqAveE} \)

Out[4]= \( \frac{90 h^4}{a^4 \text{ m}^2} \)
4-14. Determine whether or not the following pairs of operators commute.

<table>
<thead>
<tr>
<th>Õ</th>
<th>Ô</th>
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</thead>
<tbody>
<tr>
<td>(a) ( \frac{d}{dx} )</td>
<td>( \frac{d^2}{dx^2} + 2 \frac{d}{dx} )</td>
</tr>
<tr>
<td>(b) ( x )</td>
<td>( \frac{d}{dx} )</td>
</tr>
<tr>
<td>(c) SQR</td>
<td>SQRT</td>
</tr>
<tr>
<td>(d) ( x^2 \frac{d}{dx} )</td>
<td>( \frac{d^2}{dx^2} )</td>
</tr>
</tbody>
</table>

\[ a) \quad \frac{d}{dx} \left( \frac{d^2}{dx^2} + 2 \frac{d}{dx} \right) f(x) - \left( \frac{d^2}{dx^2} + 2 \frac{d}{dx} \right) \frac{d}{dx} f(x) = 0 \quad \text{Commutes} \]

\[ b) \quad \left( x^2 \frac{d}{dx} - \frac{d}{dx} x \right) f(x) = x \frac{df}{dx} - \frac{df}{dx} x f - x \frac{df}{dx} - x \frac{df}{dx} - f \neq 0 \quad \text{Does not commute} \]

\[ c) \quad (SQR \cdot SQRT - SQRT \cdot SQR) f = \]

\[ (SQR \sqrt{f} - SQRT f^2) = f - f = 0 \quad \text{Commutes if } f \text{ is real and positive} \]

\[ d) \quad \left( x^2 \frac{d}{dx} - \frac{d}{dx} x \right) f(x) \]

\[ = \left( x^2 \frac{d^3 f}{dx^3} - \frac{d}{dx} x \frac{d^2 f}{dx^2} \right) \]

\[ = \left( x^2 \frac{d^3 f}{dx^3} - \frac{d}{dx} x \frac{d^2 f}{dx^2} - x \frac{d^2 f}{dx^2} - x^2 \frac{df}{dx} \right) - \left( x \frac{d^2 f}{dx^2} + x \frac{df}{dx} \right) \]

\[ = \left( x^2 \frac{d^3 f}{dx^3} - 2 x^2 \frac{df}{dx} - x \frac{d^2 f}{dx^2} - x^2 \frac{df}{dx} - x^2 \frac{df}{dx} \right) \neq 0 \quad \text{Does not commute} \]
In ordinary algebra, \((P + Q)(P - Q) = P^2 - Q^2\). Expand \((\hat{P} + \hat{Q})(\hat{P} - \hat{Q})\). Under what conditions do we find the same result as in the case of ordinary algebra?

\[
(\hat{P} + \hat{Q})(\hat{P} - \hat{Q}) = \hat{P}^2 - \hat{P}\hat{Q} + \hat{Q}\hat{P} - \hat{Q}^2
\]

Same as "ordinary algebra" then \(\hat{P} + \hat{Q}\) commute i.e. \(\hat{P}\hat{Q} - \hat{Q}\hat{P} = 0\) or \([\hat{P}, \hat{Q}] = 0\).

Referring to Table 4.1 for the operator expressions for angular momentum, show that

\[
[L_x, \hat{L}_y] = i\hbar \hat{L}_z
\]

\[
[L_y, \hat{L}_z] = i\hbar \hat{L}_x
\]

and

\[
[L_z, \hat{L}_x] = i\hbar \hat{L}_y
\]

(Do you see a pattern here to help remember these commutation relations?) What do these expressions say about the ability to measure the components of angular momentum simultaneously?
\[
\begin{align*}
\text{Theorem: } [a, b, c] &= \alpha \left[ b, c \right] + [a, c] b & \text{add + subtract} \\
[b, c] &= \alpha \left[ b, c \right] + [a, c] b & \text{add + subtract}
\end{align*}
\]

\[
\begin{align*}
\text{proof: } [a, b, c] &= abc - cab = abc - ac b + abc - cab \\
&= a(bc - cb) + (ac - ca) b \\
&= a \left[ b, c \right] + [a, c] b
\end{align*}
\]

\[
\begin{align*}
\text{Theorems } [a, b, c] &= b \left[ a, c \right] + [a, b] c \\
\text{proof: } [a, b, c] &= abc - bca \\
&= b \left[ a, c \right] + [a, b] c = bac - bca + abc - bca = abc - bca
\end{align*}
\]

\[
\begin{align*}
\text{thus } [a, b, d] &= a \left[ b, c, d \right] + [a, c, d] b \\
&= a \left[ b, d \right] + \left[ b, c, d \right] + (c \left[ a, d \right] + [a, c] d) b \\
&= ac \left[ b, d \right] + a \left[ b, c \right] d + c \left[ a, d \right] b + c \left[ a, c \right] d b
\end{align*}
\]

\[
\begin{align*}
\text{let us work through in sequence.} \\
\text{Theorem } (a+b) \left[ c, d \right] &= \left[ a, c \right] + \left[ b, c \right] \\
\text{check } (a+b)c - c(a+b) &= ac - ca + b = cb \\
a + b c - c(\alpha + b) &= ac - ca + bc - cb \\
\end{align*}
\]

\[
\begin{align*}
\text{OK, now } [L_y, L_z] &= \left[ z p_x - x p_y, x p_y - y p_x \right] \\
&= \left[ z p_x, x p_y \right] - \left[ z p_x, y p_x \right] - \left[ x p_y, x p_y \right] + \left[ x p_y, y p_x \right] \\
&= z \left[ x p_y, p_x \right] + \left[ z, x p_y \right] p_x + x \left[ z, y p_x \right] + \left[ x, y p_x \right] p_x \\
&= z x \left[ p_y, p_x \right] + z \left[ p_x, x \right] p_y + x \left[ z, y p_x \right] + y \left[ x, p_x \right] p_z + z \left[ z, x \right] p_x p_z \\
&= z (-i k) p_y + y (i k) p_z \\
&= + i k (y p_z - z p_y) = + i k L_x
\end{align*}
\]

\[
[\chi_y, L_z] = + i k L_x
\]
4-17 (continued)

Likewise \( [L_z, L_y] = i\hbar L_x \)

Cyclic order

\[ x, y, z, \gamma \]

Since the angular momentum component operators do not commute, no two components can commute simultaneously.

\( 4-18 \) Defining

\[ L^2 = L_x^2 + L_y^2 + L_z^2 \]

show that \( L^2 \) commutes with each component separately. What does this result tell you about the ability to measure the square of the total angular momentum and its components simultaneously?

\[
[ L^2, L_x ] = [ L_x^2 + L_y^2 + L_z^2, L_x ] = [ L_x^2, L_x ] + [ L_y^2, L_x ] + [ L_z^2, L_x ] \\
= L_y [ L_y, L_x ] + L_z [ L_z, L_x ] + L_z [ L_z, L_x ] + L_z [ L_z, L_x ] L_z \\
= L_y ( -i\hbar L_z ) + L_z ( +i\hbar L_y ) + L_z ( +i\hbar L_y ) L_z \\
= +i\hbar ( -L_y L_z - L_z L_y + L_z L_y + L_y L_z ) = 0 \quad \text{G.E.D.}
\]

The long way:

\[
[ L^2, L_x ] = ( L_x^2 + L_y^2 + L_z^2 ) L_x - L_x ( L_x^2 + L_y^2 + L_z^2 ) \\
= L_x^3 + L_y^2 L_x + L_z^2 L_x - L_y^2 L_x - L_z^2 L_x \\
= i\hbar^2 \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \\
+ i\hbar^2 \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right) \\
- i\hbar^2 \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial x} \right) \\
- i\hbar^2 \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)
\]
3. assigned 9/17/02
Show that if $\psi_1(x)$ satisfies the Schrödinger equation for some potential $V(x)$ and that $\psi_2(x)$ is also a solution to the same Schrödinger equation with the same eigenvalue, then $\psi_3(x) = a\psi_1(x) + b\psi_2(x)$ is also a solution where $a$ and $b$ are arbitrary constants.

$$\mathcal{H}\psi_1(x) = E_0\psi_1(x)$$
$$\mathcal{H}\psi_2(x) = E_0\psi_2(x)$$
$$\mathcal{H}(a\psi_1 + b\psi_2) = a\mathcal{H}\psi_1 + b\mathcal{H}\psi_2 = aE_0\psi_1 + bE_0\psi_2 = E_0(a\psi_1 + b\psi_2)$$

$$a\mathcal{H}\psi_3 = E_0\psi_3 \quad \text{Q.E.D.}$$

4. assigned 9/24/02
We will be using commutators of operators quite a bit in this course. The following are two simple theorems involving commutators which will prove quite useful in doing some of the regular HW problems. First some notation.

Assume that $a$, $b$, and $c$ are operators. The notation $[a, b]$ is read the commutator of the operators $a$ and $b$ and is defined as $[a, b] = ab - ba$.

So, for example

$$[x, p_x]f(x) = (xp_x - p_xx)f(x) = xp_xf(x) - p_xxf(x) = x(-ih\partial/\partial_x)f(x) - (-ih\partial/\partial_x)xf(x) = ihf(x)$$

or $[x, p_x] = ih$

Here are the theorems I am asking you to prove:

a) prove: $[ab, c] = a[b, c] + [a, c]b$

$$[ab, c] = abc - cab \quad \text{add subtract} \quad abc :$$
$$[ab, c] = abc - abc + abc - cab$$
$$[ab, c] = a(bc - cb) + (ac - ca)b$$
$$[ab, c] = a[b, c] + [a, c]b \quad \text{Q.E.D}$$

and b) prove: $[a, bc] = b[a, c] + [a, b]c$

$$[a, bc] = abc - bca \quad \text{add subtract} \quad bac :$$
$$[a, bc] = abc - bac + bac - bca$$
$$[a, bc] = (ab - ba)c + b(ac - ca)$$
$$[a, bc] = b[a, c] + [a, b]c \quad \text{Q.E.D}$$
With these two simple theorems, more complicated combinations, such as $[a b, c d]$, can be easily expanded.

c) Use theorem (a) to show that $[x^2, p_x] = 2i\hbar x$

\[
 [a b, c d] = a [b, c d] + [a, c d] b = a 0 [b, d] + a [b, c] d + c [a, d] b + [a, c] d b
\]

\[
 [x^2, p_x] = x [c, p_x] + [x, p_x] x = x (i \hbar) + (i \hbar) x = 2i \hbar x
\]

5. assigned 9/24/02

Two operators, $A$ and $B$, commute. Prove that if $|\psi\rangle$ is an eigenfunction of $A$ with eigenvalue $a$, then $B|\psi\rangle$ is also an eigenfunction of $A$ with eigenvalue $a$.

\[
 [A, B] = 0
\]

\[
 A |\psi\rangle = a |\psi\rangle
\]

\[
 (AB - BA) |\psi\rangle = 0 \Rightarrow AB |\psi\rangle = BA |\psi\rangle
\]

\[
 AB |\psi\rangle = a B |\psi\rangle
\]

\[
 A (B |\psi\rangle) = a (B |\psi\rangle)
\]

If we call $B |\psi\rangle = |\psi\rangle$

\[
 A |\psi\rangle = a |\psi\rangle \quad QED
\]